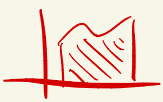
Overview

- Discord - collaboration is good! Don't cheat yourself by looking up answers
- Office Hours: TWF 12:00-1:00 PST
- Extra credit Tuesdays: worked problems & discussion
  Most EC ≤ 5% (most likely, 3%)
  - Gradescope practice survey tomorrow (Tues Aug 3, 11:25–11:40 AM)
- Computational vs Conceptual HW ← Look at schedule on Canvas
  - L completion only — UPS i
  - L graded — PS i

Today: 15.1 & 15.2

15.1: Double and iterated integrals over Rectangles

Recall Mat 21: integrals ⟺ area, and describe "accumulation"

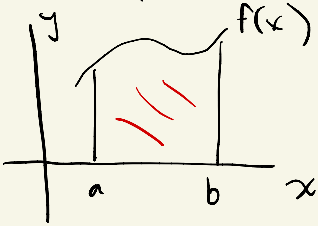


(integrated velocity $\int v\,[m/s]\,dt$ over time = position)

(integrated density $\int \rho\,[\text{stuff}/m]\,dx$ over space = mass)

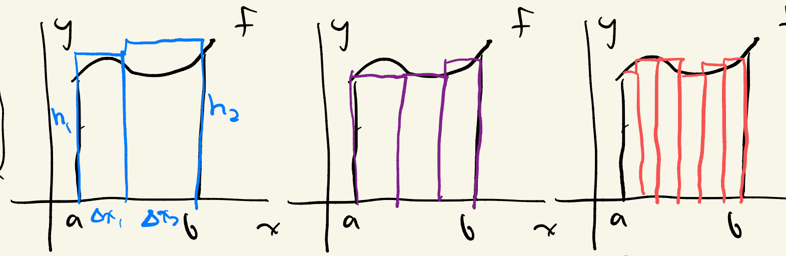
- Derivatives are useful to calculate integrals, but the definition is area under the curve
  - Limits: We can approximate numbers (like π) by convergent sequences:
    seq = (3, 3.1, 3.19, 3.14, 3.142, 3.1415, ...)

· Integrals, define this area as a limit of Riemann Sums



$$\int_a^b f(x)\,dx$$

Seq Riemann Sums:

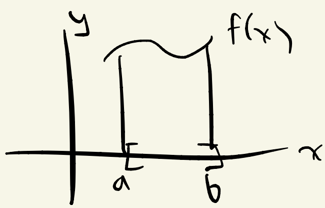


Seq: $\sum$ Areas $= \sum_{i=1}^{2} h_i \Delta x_i$, $\sum$ Areas $\sum_{i=1}^{3} h_i \Delta x_i$, $\sum$ Areas $\sum_{i=1}^{n} h_i \Delta x_i$

what matters: this sequence of areas needs to converge and the maximum width of a rectangle $\to 0$

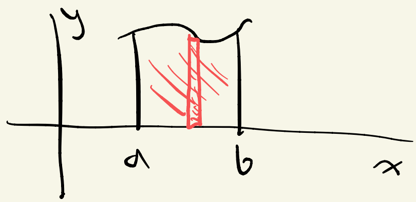
Integration

1D functions

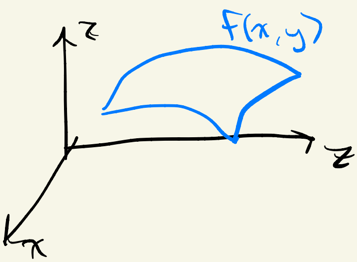


$\int (\ \ ) \, dx$ $\longmapsto$

2D area

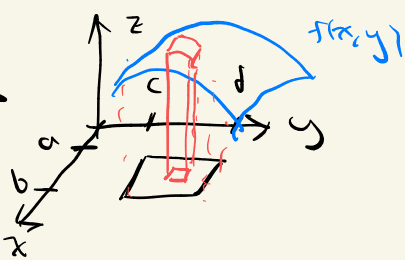


2D Functions

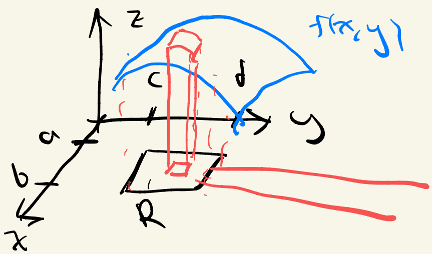


$\iint (\ \ ) \, dA$ $\longmapsto$

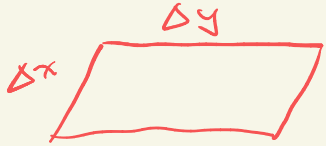
3D volume



3D volume



$\Delta \text{Volume} \approx f(x,y)\,\Delta x \Delta y$



$\Delta \text{Area} = \Delta x \Delta y$

area of rectangle

To define integral, same rule as before:

Limit of Riemann sums

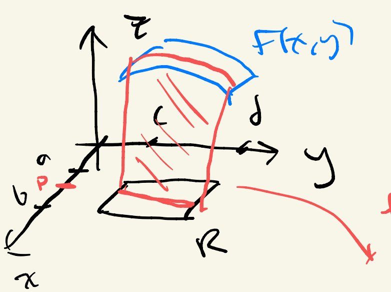
$$\iint_R f(x,y)\,dA = \lim_{[\text{Riemann sums}]} \sum \Delta \text{Vol}$$

often (rectangles), $dA = dx\,dy$.
But sometimes, dA is something else

Now: How to compute?

well... we know how to calculate volumes from cross sectional areas.

$$\text{Volume} = \int_a^b \text{Area}(x)\,dx$$

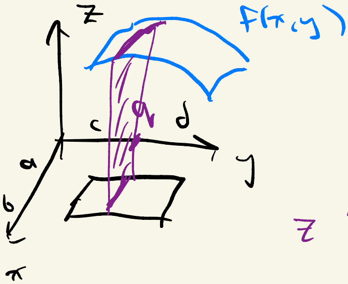


fixed $x = p$

$$\text{Area}(p) = \int_c^d f(p,y)\,dy$$

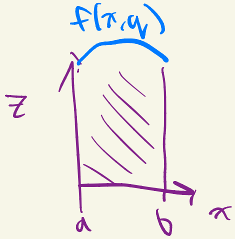
since p is fixed, f(p,y) is now a 1 variable function

But there's nothing special about fixing x and calculating $\text{Area}(x)$...



Fix $y = q$

$$\text{Vol} = \int_c^d \underbrace{\text{Area}(y)}\, dy$$

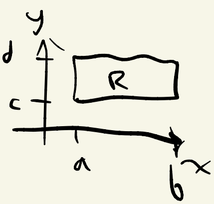


$$\text{Area}(q) = \int_a^b f(x,q)\,dx$$

The point:

Thm 1 Fubini's Thm (first form):

If $f(x,y)$ is continuous in rectangle R, $a \le x \le b$
$c \le y \le d$

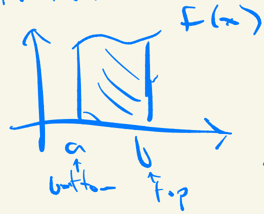


Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

10:55

Note:



$a \le b$

$$\int_{x=a}^{x=b} f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\,dx\,dy$$

Ex:

Volume bounded by elliptical parabola $z = 10 + x^2 + 3y^2$

and below by rectangle R: $0 \le x \le 1$

$0 \le y \le 2$

Soln:

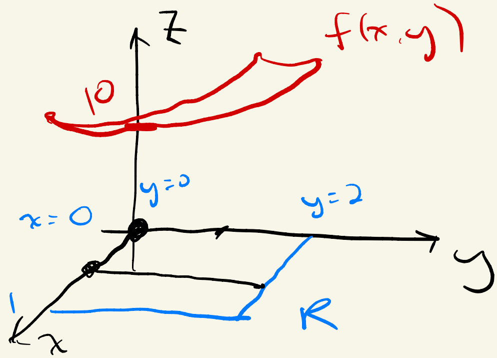
well, $f(x,y) = 10 + x^2 + 3y^2$. This a sum of polynomials.

polys are continuous, and sums of continuous functions are continuous,

so f is continuous.

use Fubini

pick x, Then y is a function of x



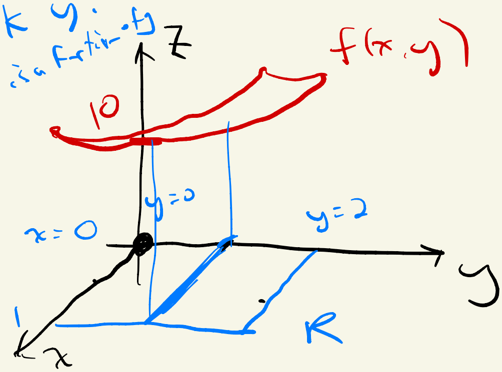
$$\text{Vol} = \int_{x=0}^{x=1}\int_{y=0}^{y=2} 10 + x^2 + 3y^2 \, dy \, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3\right]_{y=0}^{y=2} dx$$

$$= \int_0^1 20 + 2x^2 + 8 \, dx$$

$$= \frac{86}{3}$$

pick y, Then x is a function of y



$$\text{Vol} = \int_{y=0}^{y=2}\int_{x=0}^{x=1} 10 + x^2 + 3y^2 \, dx \, dy$$

$$= \int_0^2 \left[10x + \frac{x^3}{3} + 3xy^2\right]_{x=0}^{x=1} dy$$

$$= \int_0^2 10 + \frac{1}{3} + 3y^2 \, dy$$

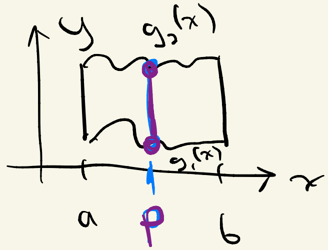
$$= \frac{86}{3}$$

15.2: Double integrals over general regions

Regions

$a \le b$

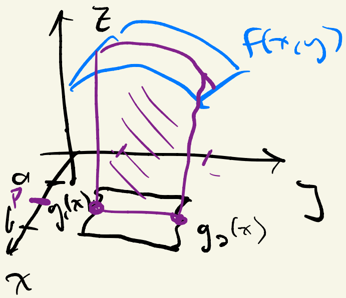
$g_1(x) \le g_2(x)$



Look: when we fix $x = p$,

we get $g_1(p)$ and $g_2(p)$, just numbers.

$g_1(p) \le g_2(p)$.



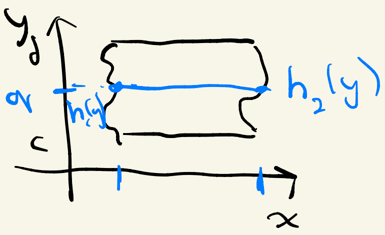
$$\text{Area}(p) = \int_{y=g_1(p)}^{y=g_2(p)} f(p,y)\,dy$$

$$\Rightarrow \text{Vol} = \int_{x=a}^{x=b} \int_{y=g_1(p)}^{y=g_2(p)} f(x,y)\,dy\,dx$$

Same for y:

$\tilde{R}$

$h_1(y) \le h_2(y)$



$$\text{Area}(q) = \int_{x=h_1(q)}^{x=h_2(q)} f(x,q)\,dx$$

$$\Rightarrow \text{Vol} = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$

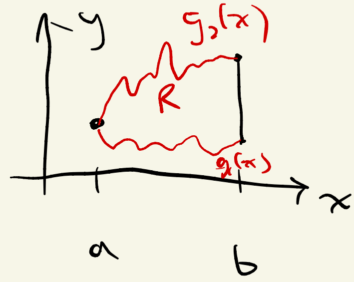
Thm 2: Stronger Fubini

$f(x,y)$ continuous on R

- Region R is $a \le x \le b$

  $g_1(x) \le y \le g_2(x)$

  with g_1, g_2 continuous on a, b

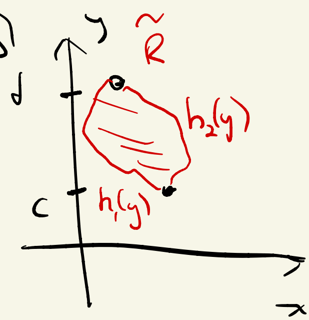


$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

- Similarly: Region $\tilde{R}$ $c \le y \le d$

  $h_1(y) \le x \le h_2(y)$



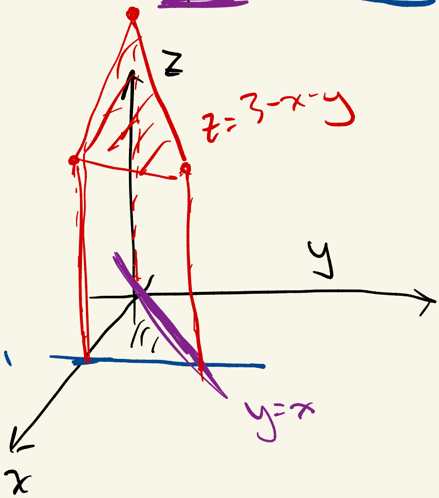
$$\iint_{\tilde{R}} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex: Volume of Prism

Base is triangle in xy plane bounded by x axis and lines $y=x$ and $x=1$. Height is the plane

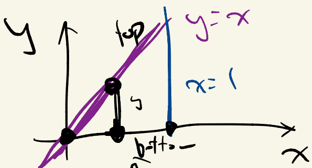
$$z = f(x,y) = 3-x-y$$

linear functions are continuous, so Fubini's applies



Option 1:

Fix x. where does y range? (making $y=g(x)$)



$$\boxed{0 \le y \le x}$$

$$\Rightarrow \text{Vol} = \int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x,y)\, dy\, dx$$

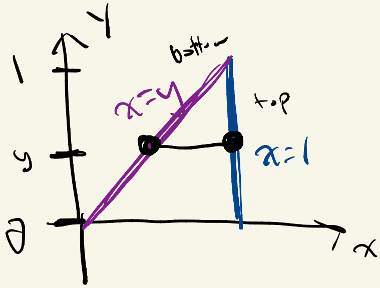
$$= \int_0^1 \int_0^x 3-x-y \; dy\, dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 3x - \frac{3x^2}{2}\, dx \quad = 1$$

· Option 2.

Fix y. Where does x range? (Find bounding functions



$y \le x \le 1$ ←

$$\Rightarrow \text{Vol} = \int_{y=0}^{y=1} \int_{x=y}^{x=1} 3 - x - y \, dx \, dy$$

$$= \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \, dy$$

$$= 1$$

Fubini worked!

Properties (Basically same as single variable)

If $f(x,y)$ and $g(x,y)$ are continuous on bounded region R

1. Linearity: α a scalar, then

$$\iint_R \alpha f(x,y) + g(x,y)\, dA = \alpha \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$\int c\,x + x^2\, dx = \int c\,x\, dx + \int x^2\, dx$$
$$= c\int x\, dx + \int x^2\, dx$$

2. Domination

If $f(x,y) \geq 0$ on R, then

$$\iint_R f(x,y)\, dA \geq 0$$

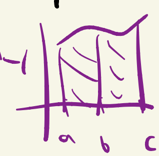
using linearity: if $\boxed{f(x,y) \geq g(x,y)}$, then $f(x,y) - g(x,y) \geq 0$,

and so

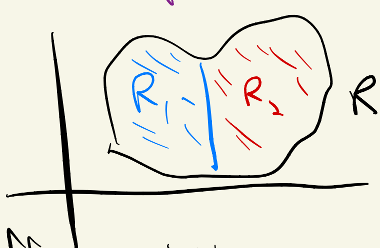
$$0 \leq \iint_R f(x,y) - g(x,y)\, dA \overset{\text{Linearity}}{=} \iint_R f(x,y)\, dA - \iint_R g(x,y)\, dA$$

$$\Rightarrow \boxed{\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA}$$

3. Additivity

Previously 

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$



If $R = R_1 \cup R_2$, then

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$